

NATURAL FROBENIUS SUBMANIFOLDS

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ABSTRACT. I.A.B. Strachan introduced the notion of a natural Frobenius submanifold of a Frobenius manifold and gave a sufficient but not necessary condition for a submanifold to be a natural Frobenius submanifold. This paper will give a necessary and sufficient condition and classify the natural Frobenius hypersurfaces.

0. INTRODUCTION

0.1. Saito structure and Frobenius manifold structure. Frobenius manifolds were introduced and investigated by B. Dubrovin as the axiomatization of a part of the rich mathematical structure of the Topological Field Theory (TFT): cf. [1, 2, 3]

A Frobenius manifold (or called Frobenius structure on M) is a quadruple $(M, \circ, \bar{g}, e, \mathcal{E})$. Here M is a manifold in one of the standard categories (C^∞ , analytic, ...), \bar{g} is a metric on M (that is, a symmetric, non-degenerate bilinear form, also denoted by \langle, \rangle), \circ is a commutative and associative product on TM and depends smoothly on M , such that if $\bar{\nabla}$ denote the Levi-Civita connection of \bar{g} , then

- a) $\bar{\nabla}$ is flat;
- b) $\bar{g}(X \circ Y, Z) = \bar{g}(X, Y \circ Z)$, for any $X, Y, Z \in TM$;
- c) the unit vector field e is covariant constant w.r.t. $\bar{\nabla}$

$$\bar{\nabla}e = 0;$$

- d) let

$$c(X, Y, Z) := \bar{g}(X \circ Y, Z)$$

(a symmetric 3-tensor). We require the 4-tensor

$$(\bar{\nabla}_Z c)(U, V, W)$$

to be symmetric in the four vector fields U, V, W, Z .

- e) A vector field \mathcal{E} must be determined on M such that

$$(0.1) \quad \bar{\nabla}(\bar{\nabla}\mathcal{E}) = 0;$$

$$(0.2) \quad \mathcal{L}_{\mathcal{E}}(\circ) = \circ;$$

$$(0.3) \quad \exists D \in \mathbb{C}, \quad \mathcal{L}_{\mathcal{E}}(\bar{g}) = D \cdot \bar{g}.$$

Remark 0.4. In this definition, because the metric \bar{g} is flat and the unit field e is covariant constant w.r.t. $\bar{\nabla}$, then (0.3) implies (0.1).

Good reference is the last chapter in [4].

There are several equivalent ways to describe a Frobenius structure. One way, called Saito structure, is recalled here:

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Definition 0.5. Let M be a complex analytic manifold of dimension m . A Saito structure on M (without metric) consists of the following data:

- 1) a flat torsion free connection $\bar{\nabla}$ on the tangent bundle TM ;
- 2) a symmetric Higgs field Φ on the tangent bundle TM , that is, Φ is an \mathcal{O}_M -linear map $\Phi: \mathcal{O}(TM) \rightarrow \Omega_M^1 \otimes \mathcal{O}(TM)$ such that

$$\Phi_X \Phi_Y = \Phi_Y \Phi_X;$$

- 3) two global sections (vector fields) e and \mathcal{E} of Θ_M , respectively called unit field and Euler field of the structure.

These data are subject to the following conditions:

- a) the meromorphic connection $\tilde{\nabla}$ on the bundle π^*TM on $\mathbb{P}^1 \times M$ defined by the formula

$$\tilde{\nabla} = \pi^*\bar{\nabla} + \frac{\pi^*\Phi}{z} - \left(\frac{\Phi(\mathcal{E})}{z} + \bar{\nabla}\mathcal{E}\right)\frac{dz}{z}$$

is integrable;

- b) the field e is $\bar{\nabla}$ -horizontal (i.e., $\bar{\nabla}e = 0$) and satisfies $\Phi_e = -\text{Id}$ (i.e., the product \circ associated to Φ has e as a unit field).

Definition 0.6. Let M be a complex analytic manifold of dimension m . A Saito structure on M with metric consists of a Saito structure $(\bar{\nabla}, \Phi, e, \mathcal{E})$ and of a metric \bar{g} on the tangent bundle, satisfying the following properties:

- (1) $\bar{\nabla}\bar{g} = 0$ (hence $\bar{\nabla}$ is the Levi-Civita connection of \bar{g});
- (2) $\Phi^* = \Phi$, i.e., for any local section X of Θ_M , $\Phi_X^* = \Phi_X$, where $*$ denotes the adjoint w.r.t. \bar{g} ;
- (3) there exists a complex number $D \in \mathbb{C}$ such that

$$\bar{\nabla}\mathcal{E} + (\bar{\nabla}\mathcal{E})^* = D \cdot \text{Id};$$

Proposition 0.7 ([1, 4]). *On any manifold M , there is an equivalence between a Saito structure with metric and a Frobenius structure.*

0.2. Frobenius submanifolds. In [5] the author considers Frobenius structures defined on open subsets of \mathbb{R}^n or \mathbb{C}^n and their (natural) Frobenius submanifolds. In [6], the author studied the submanifolds N of a semi-simple Frobenius manifold M with the Euler vector field \mathcal{E} tangent to N . We now generalize the definition of a natural Frobenius submanifold for any Frobenius manifold in the following way:

Let $(M, \bar{g}, \circ, e, \mathcal{E})$ be a Frobenius manifold, where e is the unit vector field, \mathcal{E} is the Euler vector field. Let N be a submanifold of M such that the metric \bar{g} restricted to N , denoted by g , is non-degenerate. So for any tangent vector fields $X, Y \in \Gamma(U, TN)$ we can define a new product in TN by $X * Y := \text{pr}(X \circ Y)$. Similarly we set $e_N := \text{pr}(e)$, $\mathcal{E}_N := \text{pr}(\mathcal{E})$, where $\text{pr}: TM \rightarrow TN$ is the orthogonal projection on TN w.r.t. \bar{g} . We set $TN^\perp = \{\xi \in TM \mid \forall X \in TN, \langle X, \xi \rangle = 0\}$. So for any vector field $X \in TM$, we have the decomposition:

$$X = \text{pr}(X) + X^\perp,$$

where $\text{pr}(X) \in TN$, $X^\perp \in TN^\perp$.

Definition 0.8. The submanifold N is called a *Frobenius submanifold* of the Frobenius manifold $(M, \bar{g}, \circ, e, \mathcal{E})$ if the induced structure $(N, g, *, e_N, \mathcal{E}_N)$ on N is a Frobenius manifold structure.

Definition 0.9. The Frobenius submanifold N of $(M, \bar{g}, \circ, e, \mathcal{E})$ is called *natural* if TN is left invariant by the product \circ .

In [5] the author gave a sufficient condition for a submanifold N to be a natural Frobenius submanifold:

Theorem 0.10 ([5]). *Let N be a flat submanifold of a Frobenius manifold M with*

$$\begin{aligned} e|_N &\in TN; \\ TN \circ TN &\subseteq TN; \\ \mathcal{E}|_N &\in TN. \end{aligned}$$

Then N is a natural Frobenius submanifold

Neither $e|_N \in TN$ nor $\mathcal{E}|_N \in TN$ is necessary, we will construct examples of natural Frobenius submanifolds such that $e|_N \neq e_N$ and $\mathcal{E}|_N \neq \mathcal{E}_N$.

Example 0.11. Let $(N, g, *, e_N, \mathcal{E}_N)$ to be a Frobenius manifold of dimension n with constant $D \neq 0$, and let \mathcal{A} be the affine line. Define a new Frobenius manifold $M = N \times \mathcal{A}$ as follows:

Let z be the coordinate of \mathcal{A} and choose a metric η on \mathcal{A} such that $\eta(\partial_z, \partial_z) = 1$. We define a new metric \bar{g} on M to be the direct sum of g and η . Let $\bar{\nabla}$ be the Levi-Civita connection of \bar{g} . Then $\bar{\nabla}$ is just the direct sum of ∇ and d , where d is the Levi-Civita connection of η . Now define a product \circ : For any $X, Y \in TN$,

$$\begin{aligned} X \circ Y &:= X * Y; \\ X \circ \partial_z &= 0; \\ \partial_z \circ \partial_z &= \frac{2}{D} \partial_z. \end{aligned}$$

Finally we define the unit element e and the Euler vector field \mathcal{E} :

$$\begin{aligned} e &= e_N + \frac{D}{2} \partial_z; \\ \mathcal{E} &= \mathcal{E}_N + \frac{D}{2} z \partial_z. \end{aligned}$$

It is easy to see $(M, \bar{g}, \circ, e, \mathcal{E})$ is a Frobenius manifold. Now we embed N to M :

$$\iota : N \longrightarrow M, \quad P \longmapsto (P, 1).$$

Then we get a natural Frobenius submanifold $N \times \{1\}$ with $e_N^\perp \neq 0$ and $\mathcal{E}_N^\perp \neq 0$.

0.3. Aim of the paper. The paper will give a necessary and sufficient condition for a submanifold to be a natural Frobenius manifold and classify the natural Frobenius hypersurfaces.

Let us first recall a known result in differential geometry which will explain the notation in results below:

Theorem 0.12 ([7]). *Let M be a manifold with a metric \bar{g} , $\bar{\nabla}$ is the Levi-Civita connection of \bar{g} , and let N be an arbitrary submanifold such that the restricted metric g is non-degenerate. Then for all $W, X, Y, Z \in TN$ and normal vectors $\xi, \eta \in TN^\perp$, w.r.t. the decomposition $TN \oplus TN^\perp$, we have:*

Gauss formula:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

Weingarten formula:

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where $\nabla_X Y := \text{pr}(\bar{\nabla}_X Y)$, $-A_\xi X := \text{pr}(\bar{\nabla}_X \xi)$.

Here h is called the second fundamental form and A is called the shape operator, which are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle,$$

for any $X, Y \in TN$, $\xi \in TN^\perp$.

We have the following result for any submanifold.

Theorem 0.13. *Let $(M, \bar{g}, \circ, e, \mathcal{E})$ be a Frobenius manifold. Let N be a submanifold of M such that $g := \bar{g}|_{TN}$ is non-degenerate and flat. Then the following properties are equivalent:*

- (1) N is a natural Frobenius submanifold of M ;
- (2)

$$(0.14) \quad \nabla e_N = 0;$$

$$(0.15) \quad TN \circ TN \subseteq TN;$$

$$(0.16) \quad \exists \lambda \in \mathbb{C}, \quad A_{\mathcal{E}_N^\perp} = \lambda \cdot \text{Id},$$

We classify the natural Frobenius hypersurfaces in the following:

Proposition 0.17. *With the assumptions of Theorem 0.13, assume moreover that N is a hypersurface.*

(a) *If e is not tangent to N , then the following are equivalent:*

- (1) N is natural Frobenius submanifold of M ;
- (2) $TN \circ TN \subseteq TN, \bar{\nabla} = \nabla$.

(b) *If e is tangent to N , then the following are equivalent:*

- (1) N is natural Frobenius submanifold of M ;
- (2) $TN \circ TN \subseteq TN$ and either $\bar{\nabla} = \nabla$ or \mathcal{E} is tangent to N .

We will give some examples in the last section to show that this classification can not be generalized to all submanifolds.

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1. GENERAL DIMENSION

In this section we mainly give a necessary and sufficient condition for a submanifold to be a natural Frobenius manifold. Because of the equivalence between Frobenius structure and Saito structure with metric, we will see the sufficient condition from these two equivalent point of view.

Lemma 1.1. *Let $(M, \bar{g}, \circ, e, \mathcal{E})$ be a Frobenius manifold. Let N be a submanifold of M such that $TN \circ TN \subseteq TN$, then $TN \circ TN^\perp \subseteq TN^\perp$.*

The proof of Lemma 1.1. Because $(M, \bar{g}, \circ, e, \mathcal{E})$ is a Frobenius manifold, so we have the relation

$$\langle U \circ V, W \rangle = \langle U, V \circ W \rangle,$$

for all $U, V, W \in TM$. If $TN \circ TN \subseteq TN$, then $\forall U, V \in TN, \zeta \in TN^\perp$ we have

$$\langle U \circ \zeta, V \rangle = \langle \zeta, U \circ V \rangle = 0.$$

So $TN \circ TN^\perp \subseteq TN^\perp$. □

Proof of Theorem A. (1) \implies (2). Because N is a Frobenius submanifold of M , there exist two constants D and D_N , such that:

$$\bar{\nabla} \mathcal{E} + (\bar{\nabla} \mathcal{E})^* = D \cdot \text{Id};$$

$$\nabla \mathcal{E}_N + (\nabla \mathcal{E}_N)^* = D_N \cdot \text{Id}.$$

For any $U, V \in TN$, we have:

$$\langle \bar{\nabla}_U \mathcal{E}, V \rangle + \langle \bar{\nabla}_V \mathcal{E}, U \rangle = D \cdot \langle U, V \rangle$$

Computing the left hand side of the above equality we get:

$$\begin{aligned}
\langle \bar{\nabla}_U \mathcal{E}, V \rangle + \langle \bar{\nabla}_V \mathcal{E}, U \rangle &= \langle \bar{\nabla}_U \mathcal{E}_N + \bar{\nabla}_U \mathcal{E}_N^\perp, V \rangle + \langle \bar{\nabla}_V \mathcal{E}_N + \bar{\nabla}_V \mathcal{E}_N^\perp, U \rangle \\
&= \langle \nabla_U \mathcal{E}_N + \nabla_U^\perp \mathcal{E}_N^\perp + h(U, \mathcal{E}_N) - A_{\mathcal{E}_N^\perp} U, V \rangle \\
&\quad + \langle \nabla_V \mathcal{E}_N + \nabla_V^\perp \mathcal{E}_N^\perp + h(V, \mathcal{E}_N) - A_{\mathcal{E}_N^\perp} V, U \rangle \\
&= \langle \nabla_U \mathcal{E}_N - A_{\mathcal{E}_N^\perp} U, V \rangle + \langle \nabla_V \mathcal{E}_N - A_{\mathcal{E}_N^\perp} V, U \rangle \\
&= \langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle - \langle A_{\mathcal{E}_N^\perp} U, V \rangle - \langle A_{\mathcal{E}_N^\perp} V, U \rangle \\
&= \langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle - 2\langle h(U, V), \mathcal{E}_N^\perp \rangle \\
&= D_N \cdot \langle U, V \rangle - 2\langle h(U, V), \mathcal{E}_N^\perp \rangle \\
&= D \cdot \langle U, V \rangle
\end{aligned}$$

So we get:

$$\langle h(U, V), \mathcal{E}_N^\perp \rangle = \frac{D_N - D}{2} \cdot \langle U, V \rangle,$$

i.e.,

$$\langle A_{\mathcal{E}_N^\perp} U, V \rangle = \frac{D_N - D}{2} \cdot \langle U, V \rangle.$$

So

$$A_{\mathcal{E}_N^\perp} = \frac{D_N - D}{2} \cdot \text{Id}$$

The other two equalities $\nabla e_N = 0$ and $TN \circ TN \subseteq TN$ hold because N is the natural Frobenius submanifold of M .

(2) \implies (1)

We will give two methods to prove the sufficient condition. In the first method, we use the flat holomorphic local coordinates to prove that Condition (2) induces a Saito structure with metric on M . The most difficult part in this method is the flatness of the structure connection $\tilde{\nabla}$. The second method is more global. We prove that Condition (2) induces a Frobenius structure on M . In this method every thing is more obvious except the relation $\mathcal{L}_{\mathcal{E}_N}(\circ) = \circ$.

First method: Saito structure. Suppose $TN \circ TN \subseteq TN$ and there exists a constant $\lambda \in \mathbb{C}$ such that $A_{\mathcal{E}_N^\perp} = \lambda \cdot \text{Id}$. So the restricted Higgs field $\Phi|_{TN}$ is a Higgs field on TN , where Φ is defined by $\Phi_X Y := -X \circ Y$, for any $X, Y \in TM$.

The structure connections $\tilde{\nabla}$ on $M \times \mathbf{P}^1$ and $\tilde{\nabla}$ on $N \times \mathbf{P}^1$ are defined by

$$\begin{aligned}
\tilde{\nabla} &:= \pi^* \bar{\nabla} + \frac{\pi^* \Phi}{z} - \left(\frac{\Phi(\mathcal{E})}{z} + \bar{\nabla} \mathcal{E} \right) \frac{dz}{z}; \\
\tilde{\nabla} &:= \pi^* \nabla + \frac{\pi^* (\Phi|_{TN})}{z} - \left(\frac{\Phi(\mathcal{E}_N)}{z} + \nabla \mathcal{E}_N \right) \frac{dz}{z};
\end{aligned}$$

We will show that the induced structure $(\nabla, \Phi|_{TN}, e_N, \mathcal{E}_N, g)$ on N is a Saito structure with metric.

S1) *Existence of flat unit field.* Because of $TN \circ TN \subseteq TN$. From Lemma 1.1, we know $TN \circ TN^\perp \subseteq TN^\perp$, so for any $U \in TN$, we have:

$$U = U \circ e = U \circ e_N + U \circ e_N^\perp.$$

So

$$U \circ e_N^\perp = U - U \circ e_N \in TN \cap TN^\perp = \{0\}.$$

So

$$U = U \circ e_N,$$

for any $U \in TN$. i.e., $\Phi_{e_N}|_{TN} = -\text{Id}$. $\nabla e_N = 0$ show that the unit vector field e_N is ∇ -flat.

$S2)$ flatness of the structure connection $\tilde{\nabla}$. Denote by $\tilde{\mathcal{R}}$ the curvature of $\tilde{\nabla}$ and by $\tilde{\mathcal{R}}$ the curvature of $\tilde{\nabla}$. Because M is Frobenius manifold, $\tilde{\mathcal{R}} = 0$. For any $U, V, W \in TM$, we have:

$$\tilde{\mathcal{R}}(U, V)W = 0.$$

Computing the left hand side of the above equality:

$$\begin{aligned} \tilde{\mathcal{R}}(U, V)W &= \overline{R}(U, V)W \\ &+ \frac{1}{z}\{U \circ \overline{\nabla}_V W - V \circ \overline{\nabla}_U W - [U, V] \circ W + \overline{\nabla}_U(V \circ W) - \overline{\nabla}_V(U \circ W)\} \\ &+ \frac{1}{z^2}\{U \circ (V \circ W) - V \circ (U \circ W)\}, \end{aligned}$$

where \overline{R} is the curvature of $\overline{\nabla}$. So we get:

$$(1.2) \quad U \circ \overline{\nabla}_V W - V \circ \overline{\nabla}_U W - [U, V] \circ W + \overline{\nabla}_U(V \circ W) - \overline{\nabla}_V(U \circ W) = 0.$$

because of Lemma 1.1 we have, for $\forall U, V, W \in TN$

$$\begin{aligned} &\text{pr}\{U \circ \overline{\nabla}_V W - V \circ \overline{\nabla}_U W - [U, V] \circ W + \overline{\nabla}_U(V \circ W) - \overline{\nabla}_V(U \circ W)\} \\ &= U \circ \nabla_V W - V \circ \nabla_U W - [U, V] \circ W + \nabla_U(V \circ W) - \nabla_V(U \circ W) \\ &= 0. \end{aligned}$$

However,

$$\begin{aligned} \tilde{\mathcal{R}}(U, V)W &= R(U, V)W \\ &+ \frac{1}{z}\{U \circ \nabla_V W - V \circ \nabla_U W - [U, V] \circ W + \nabla_U(V \circ W) - \nabla_V(U \circ W)\} \\ &+ \frac{1}{z^2}\{U \circ (V \circ W) - V \circ (U \circ W)\}, \end{aligned}$$

where R is the curvature of ∇ . So

$$\tilde{\mathcal{R}}(U, V)W = 0,$$

for any $U, V, W \in TN$

Now the only other equality to be checked is $\tilde{\mathcal{R}}(z \frac{d}{dz}, U)V = 0$. Calculating directly, we get:

$$\tilde{\mathcal{R}}(z \frac{d}{dz}, U)V = -\nabla_U \nabla_V \mathcal{E}_N + \nabla_{\nabla_U V} \mathcal{E}_N.$$

for any $U, V \in TN$.

Suppose t^1, t^2, \dots, t^m is the flat coordinate of $(M, \overline{\nabla})$, $\tau^1, \tau^2, \dots, \tau^n$ is the flat coordinate of N , $\tilde{\mathcal{R}}$ is a tensor, so we just check it for base elements ∂_α . So we just need to check:

$$\partial_{\tau^\alpha} \partial_{\tau^\beta} \mathcal{E}_N^\gamma = 0,$$

for all $\alpha, \beta, \gamma \in \{1, 2, \dots, n\}$.

Locally, $\mathcal{E} = \mathcal{E}^i \partial_{t^i}$, $\mathcal{E}_N = \mathcal{E}_N^\alpha \partial_{\tau^\alpha}$. Choose the local frame of TN^\perp , denoted by $\partial_{\nu^{\tilde{\alpha}}}$, such that

$$\partial_{t^i} = A_i^\alpha \partial_{\tau^\alpha} + n_i^{\tilde{\alpha}} \partial_{\nu^{\tilde{\alpha}}}.$$

and

$$\langle \partial_{\nu^{\tilde{\alpha}}}, \partial_{\nu^{\tilde{\beta}}} \rangle = \eta_{\tilde{\alpha}\tilde{\beta}}$$

where $\eta_{\tilde{\alpha}\tilde{\beta}}$ are constant with $\eta_{\tilde{\alpha}\tilde{\beta}} = \epsilon(\tilde{\alpha})\delta_{\tilde{\alpha}\tilde{\beta}}$ with $\epsilon(\tilde{\alpha}) = \pm 1$.

Using the metrics \overline{g} and g we get:

$$A_i^\alpha = \overline{g}_{ij} g^{\alpha\beta} \frac{\partial t^j}{\partial \tau^\beta}$$

From the definition of \mathcal{E}_N , we get $\mathcal{E}_N^\alpha = \mathcal{E}^i|_N A_i^\alpha$.

Computing $\frac{\partial \mathcal{E}_N^\gamma}{\partial \tau^\beta}$ directly we get:

$$\frac{\partial \mathcal{E}_N^\gamma}{\partial \tau^\beta} = \frac{\partial \mathcal{E}^i}{\partial \tau^\beta} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial t^j}{\partial \tau^\delta} + \mathcal{E}^i \bar{g}_{ij} g^{\gamma\delta} \frac{\partial^2 t^j}{\partial \tau^\beta \partial \tau^\delta}$$

On the other hand,

$$\begin{aligned} \mathcal{E}^i \bar{g}_{ij} g^{\gamma\delta} \frac{\partial^2 t^j}{\partial \tau^\beta \partial \tau^\delta} &= \mathcal{E}^i \langle \partial_{t^i}, \partial_{t^j} \rangle g^{\gamma\delta} \frac{\partial^2 t^j}{\partial \tau^\beta \partial \tau^\delta} \\ &= \langle \mathcal{E}^i \partial_{t^i}, \frac{\partial^2 t^j}{\partial \tau^\beta \partial \tau^\delta} \partial_{t^j} \rangle g^{\gamma\delta} \\ &= g^{\gamma\delta} \langle \mathcal{E}, \bar{\nabla}_{\partial_{\tau^\beta}} \partial_{\tau^\delta} \rangle \\ &= g^{\gamma\delta} \langle \mathcal{E}, \nabla_{\partial_{\tau^\beta}} \partial_{\tau^\delta} + h(\partial_{\tau^\beta}, \partial_{\tau^\delta}) \rangle \end{aligned}$$

But $\tau^1, \tau^2, \dots, \tau^n$ is the flat coordinate of N , so

$$\begin{aligned} \mathcal{E}^i \bar{g}_{ij} g^{\gamma\delta} \frac{\partial^2 t^j}{\partial \tau^\beta \partial \tau^\delta} &= g^{\gamma\delta} \langle \mathcal{E}, h(\partial_{\tau^\beta}, \partial_{\tau^\delta}) \rangle \\ &= g^{\gamma\delta} \langle \mathcal{E}_N^\perp, h(\partial_{\tau^\beta}, \partial_{\tau^\delta}) \rangle \\ &= g^{\gamma\delta} \langle A_{\mathcal{E}_N^\perp} \partial_{\tau^\beta}, \partial_{\tau^\delta} \rangle \\ &= \lambda g^{\gamma\delta} g_{\delta\beta}. \end{aligned}$$

The last equality holds because $A_{\mathcal{E}_N^\perp} = \lambda \cdot \text{Id}$.

So we have:

$$\frac{\partial \mathcal{E}_N^\gamma}{\partial \tau^\beta} = \frac{\partial \mathcal{E}^i}{\partial \tau^\beta} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial t^j}{\partial \tau^\delta} + \lambda g^{\gamma\delta} g_{\delta\beta}$$

Similarly computing we get:

$$\frac{\partial^2 \mathcal{E}_N^\gamma}{\partial \tau^\alpha \partial \tau^\beta} = \frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial t^j}{\partial \tau^\delta} + \frac{\partial^2 t^j}{\partial \tau^\alpha \partial \tau^\delta} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial \mathcal{E}^i}{\partial \tau^\beta}$$

We will prove that the right hand side of the above equality vanishes.

Calculating the first term of the right hand side we get:

$$\begin{aligned} \frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial t^j}{\partial \tau^\delta} &= \frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \langle \partial_{t^i}, \partial_{t^j} \rangle g^{\gamma\delta} \frac{\partial t^j}{\partial \tau^\delta} \\ &= \left\langle \frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \partial_{t^i}, \frac{\partial t^j}{\partial \tau^\delta} \partial_{t^j} \right\rangle g^{\gamma\delta}. \end{aligned}$$

Claim: $\bar{\nabla}_{h(\partial_{\tau^\alpha}, \partial_{\tau^\beta})} \mathcal{E} = \frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \partial_{t^i}$

In fact, because M is a Frobenius manifold, we have $\tilde{\mathcal{R}} = 0$, so we have

$$\tilde{\mathcal{R}}(z \frac{d}{dz}, U) V = -\bar{\nabla}_U \bar{\nabla}_V \mathcal{E} + \bar{\nabla}_{\bar{\nabla}_U V} \mathcal{E} = 0,$$

for all $U, V \in TN$. By this equality we get:

$$\begin{aligned} \bar{\nabla}_{h(\partial_{\tau^\alpha}, \partial_{\tau^\beta})} \mathcal{E} &= \bar{\nabla}_{\bar{\nabla}_{\partial_{\tau^\alpha}} \partial_{\tau^\beta}} \mathcal{E} \\ &= \bar{\nabla}_{\partial_{\tau^\alpha}} \bar{\nabla}_{\partial_{\tau^\beta}} \mathcal{E} \\ &= \frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \partial_{t^i}. \end{aligned}$$

So

$$\frac{\partial^2 \mathcal{E}^i}{\partial \tau^\alpha \partial \tau^\beta} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial t^j}{\partial \tau^\delta} = \frac{\partial^2 \mathcal{E}_N^\gamma}{\partial \tau^\alpha \partial \tau^\beta} = \langle \bar{\nabla}_{h(\partial_{\tau^\alpha}, \partial_{\tau^\beta})} \mathcal{E}, \partial_{\tau^\delta} \rangle g^{\gamma\delta}$$

Similarly for the second term:

$$\frac{\partial^2 t^j}{\partial_{\tau^\alpha} \partial_{\tau^\delta}} \bar{g}_{ij} g^{\gamma\delta} \frac{\partial \mathcal{E}^i}{\partial_{\tau^\delta}} = \langle \bar{\nabla}_{\partial_{\tau^\beta}} \mathcal{E}, h(\partial_{\tau^\delta}, \partial_{\tau^\alpha}) \rangle g^{\gamma\delta}$$

We simplify the equality to be:

$$\frac{\partial^2 \mathcal{E}_N^\gamma}{\partial_{\tau^\alpha} \partial_{\tau^\beta}} = \langle \bar{\nabla}_{h(\partial_{\tau^\alpha}, \partial_{\tau^\beta})} \mathcal{E}, \partial_{\tau^\delta} \rangle g^{\gamma\delta} + \langle \bar{\nabla}_{\partial_{\tau^\beta}} \mathcal{E}, h(\partial_{\tau^\delta}, \partial_{\tau^\alpha}) \rangle g^{\gamma\delta}$$

That M is a Frobenius manifold also implies that there exists a constant D such that:

$$\bar{\nabla} \mathcal{E} + (\bar{\nabla} \mathcal{E})^* = D \cdot \text{Id};$$

So

$$\begin{aligned} \langle \bar{\nabla}_{h(\partial_{\tau^\alpha}, \partial_{\tau^\beta})} \mathcal{E}, \partial_{\tau^\delta} \rangle &= D \cdot \langle h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \partial_{\tau^\delta} \rangle - \langle h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \bar{\nabla}_{\partial_{\tau^\delta}} \mathcal{E} \rangle \\ &= -\langle h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \bar{\nabla}_{\partial_{\tau^\delta}} \mathcal{E} \rangle. \end{aligned}$$

Because $\bar{\nabla} \bar{g} = 0$, we get:

$$\begin{aligned} -\langle h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \bar{\nabla}_{\partial_{\tau^\delta}} \mathcal{E} \rangle &= -\partial_{\tau^\delta} (\langle h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \mathcal{E} \rangle) + \langle \bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \mathcal{E} \rangle \\ &= -\partial_{\tau^\delta} (\langle h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \mathcal{E}_N^\perp \rangle) + \langle \bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \mathcal{E} \rangle \\ &= -\partial_{\tau^\delta} (\lambda g_{\alpha\beta}) + \langle \bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \mathcal{E} \rangle \\ &= \langle \bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}), \mathcal{E} \rangle. \end{aligned}$$

Similarly we get:

$$\langle \bar{\nabla}_{\partial_{\tau^\beta}} \mathcal{E}, h(\partial_{\tau^\delta}, \partial_{\tau^\alpha}) \rangle = -\langle \bar{\nabla}_{\partial_{\tau^\beta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\delta}), \mathcal{E} \rangle$$

Then we have the equality:

$$\frac{\partial^2 \mathcal{E}_N^\gamma}{\partial_{\tau^\alpha} \partial_{\tau^\beta}} = g^{\delta\gamma} \langle \mathcal{E}, \bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}) - \bar{\nabla}_{\partial_{\tau^\beta}} h(\partial_{\tau^\delta}, \partial_{\tau^\alpha}) \rangle$$

However the right hand side of this equality vanishes because $\bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta})$ is totally symmetric in α, β, δ :

$$\begin{aligned} \bar{\nabla}_{\partial_{\tau^\delta}} h(\partial_{\tau^\alpha}, \partial_{\tau^\beta}) &= \bar{\nabla}_{\partial_{\tau^\delta}} \bar{\nabla}_{\partial_{\tau^\beta}} \partial_{\tau^\alpha} \\ &= \bar{\nabla}_{\partial_{\tau^\beta}} \bar{\nabla}_{\partial_{\tau^\delta}} \partial_{\tau^\alpha} + \bar{\nabla}_{[\partial_{\tau^\delta}, \partial_{\tau^\beta}]} \partial_{\tau^\alpha} \\ &= \bar{\nabla}_{\partial_{\tau^\beta}} h(\partial_{\tau^\delta}, \partial_{\tau^\alpha}). \end{aligned}$$

Then we get $\partial_{\tau^\alpha} \partial_{\tau^\beta} \mathcal{E}_N^\gamma = 0, \forall \alpha, \beta, \gamma \in \{1, 2, \dots, n\}$. So $\tilde{\mathcal{R}} = 0$, i.e., the structure connection $\tilde{\nabla}$ is integrable.

From S1) and S2) we get $(\nabla, \Phi|_{TN}, e_N, \mathcal{E}_N)$ is a Saito structure (without metric) on N .

S3) *Saito structure* $(\nabla, \Phi|_{TN}, e_N, \mathcal{E}_N)$ with metric g . Because ∇ is the Levi-Civita connection of g , so we have:

$$\nabla g = 0.$$

The induced Higgs field $\Phi|_{TN}$ satisfies $\Phi|_{TN} = (\Phi|_{TN})^*$ w.r.t. g because $\Phi = (\Phi)^*$ w.r.t. \bar{g} . So we just need to check:

$\exists D_N \in \mathbb{C}$, such that

$$\nabla \mathcal{E}_N + (\nabla \mathcal{E}_N)^* = D_N \cdot \text{Id}.$$

Because M is Frobenius manifold, there exists a constant D such that:

$$\bar{\nabla} \mathcal{E} + (\bar{\nabla} \mathcal{E})^* = D \cdot \text{Id}.$$

Computing the left hand side of the above relation as in the proof of (1) \Rightarrow (2), we get for any $U, V \in TN$

$$\begin{aligned} \langle \bar{\nabla}_U \mathcal{E}, V \rangle + \langle \bar{\nabla}_V \mathcal{E}, U \rangle &= \langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle - \langle A_{\mathcal{E}_N^\perp} U, V \rangle \\ &\quad - \langle A_{\mathcal{E}_N^\perp} V, U \rangle \\ &= \langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle - 2\lambda \cdot \langle U, V \rangle \\ &= D \cdot \langle U, V \rangle \end{aligned}$$

so

$$\langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle = D \cdot \langle U, V \rangle + 2\lambda \cdot \langle U, V \rangle$$

That is to say:

$$\nabla \mathcal{E}_N + (\nabla \mathcal{E}_N)^* = (D + 2\lambda) \cdot \text{Id}$$

Take $D_N = D + 2\lambda$, we get the equality:

$$\nabla \mathcal{E}_N + (\nabla \mathcal{E}_N)^* = D_N \cdot \text{Id}.$$

From $S1), S2), S3)$ we know that $(\nabla, \Phi|_{TN}, e_N, \mathcal{E}_N, g)$ is a Saito structure on N .

Second method: Frobenius manifold structure. Consider the quadruple $(N, \circ, g, e_N, \mathcal{E}_N)$.

$F1)$ From the assumption we know that g is flat. Just like the proof of $S1)$, we get the unit vector field e_N is ∇ -flat.

$F2)$ In other hand, $TN \circ TN \subseteq TN$ also implies that

$$\langle U \circ V, W \rangle = \langle U, V \circ W \rangle, \forall U, V, W \in TN.$$

$F3)$ Now define a new 3-tensor

$$c_N(U, V, W) := \langle U \circ V, W \rangle.$$

It is easy to see that c_N is the restricted tensor of c to $TN \otimes TN \otimes TN$, where

$$c(\bar{U}, \bar{V}, \bar{W}) := \langle \bar{U} \circ \bar{V}, \bar{W} \rangle, \forall \bar{U}, \bar{V}, \bar{W} \in TM.$$

M is a Frobenius manifold, so the 4-tensor $(\bar{\nabla}_{\bar{W}'} c)(\bar{U}, \bar{V}, \bar{W})$ is symmetric in the four vector fields $\bar{U}, \bar{V}, \bar{W}, \bar{W}' \in TM$.

So for any $U, V, W, W' \in TN$ we have

$$\begin{aligned} (\bar{\nabla}_{W'} c)(U, V, W) &= W' (c(U, V, W)) - c(\bar{\nabla}_{W'} U, V, W) - c(U, \bar{\nabla}_{W'} V, W) - c(U, v, \bar{\nabla}_{W'} W) \\ &= W' (c_N(U, V, W)) - c(\nabla_{W'} U, V, W) - c(U, \nabla_{W'} V, W) - c(U, v, \nabla_{W'} W) \\ &\quad - c(h(W', U), V, W) - c(U, h(W', V), W) - c(U, v, h(W', W)) \end{aligned}$$

However for any $U, V, W, W' \in TN$ we have

$$c(h(W', U), V, W) = \langle h(W', U) \circ V, W \rangle = \langle h(W', U), W \circ V \rangle = 0.$$

So for any $U, V, W, W' \in TN$ we get

$$\begin{aligned} (\bar{\nabla}_{W'} c)(U, V, W) &= W' (c_N(U, V, W)) - c(\nabla_{W'} U, V, W) - c(U, \nabla_{W'} V, W) - c(U, v, \nabla_{W'} W) \\ &= W' (c_N(U, V, W)) - c_N(\nabla_{W'} U, V, W) - c_N(U, \nabla_{W'} V, W) - c_N(U, v, \nabla_{W'} W) \\ &= (\nabla_{W'} c_N)(U, V, W). \end{aligned}$$

But we know that the 4-tensor $(\bar{\nabla}_{\bar{W}'}c)(\bar{U}, \bar{V}, \bar{W})$ is symmetric in the four vector fields $\bar{U}, \bar{V}, \bar{W}, \bar{W}' \in TM$. Specially, it is symmetric in the four vector fields $U, V, W, W' \in TN$, i.e., we get

$$(\nabla_{W'}c_N)(U, V, W)$$

is symmetric in the four vector fields $U, V, W, W' \in TN$.

F4) Now consider the vector field \mathcal{E}_N

M is a Frobenius manifold, so there exists a constant D such that

$$\bar{\nabla}\mathcal{E} + (\bar{\nabla}\mathcal{E})^* = D \cdot \text{Id}.$$

Computing the left hand side of the above relation as in (1) \Rightarrow (2), together with the condition $A_{\mathcal{E}_N^\perp} = \lambda \cdot \text{Id}$, we have for any $U, V \in TN$

$$\begin{aligned} \langle \bar{\nabla}_U \mathcal{E}, V \rangle + \langle \bar{\nabla}_V \mathcal{E}, U \rangle &= \langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle - \langle A_{\mathcal{E}_N^\perp} U, V \rangle - \langle A_{\mathcal{E}_N^\perp} V, U \rangle \\ &= \langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle - 2\lambda \cdot \langle U, V \rangle \\ &= D \cdot \langle U, V \rangle, \end{aligned}$$

so

$$\langle \nabla_U \mathcal{E}_N, V \rangle + \langle \nabla_V \mathcal{E}_N, U \rangle = D \cdot \langle U, V \rangle + 2\lambda \cdot \langle U, V \rangle$$

That is to say:

$$\nabla \mathcal{E}_N + (\nabla \mathcal{E}_N)^* = (D + 2\lambda) \cdot \text{Id}$$

Take $D_N = D + 2\lambda$, we get the equality:

$$\nabla \mathcal{E}_N + (\nabla \mathcal{E}_N)^* = D_N \cdot \text{Id}.$$

Because g is flat, this relation is equivalent to

$$\mathcal{L}_{\mathcal{E}_N}(g) = D_N \cdot g.$$

F5) We will prove

$$\mathcal{L}_{\mathcal{E}_N}(\circ) = \circ.$$

Modulo the relation $\nabla(\Phi|_{TN}) = 0$ this is equivalent to the relation:

$$\nabla_U(V \circ \mathcal{E}_N) - (\nabla_U V) \circ \mathcal{E}_N + U \circ \nabla_V \mathcal{E}_N - \nabla_{U \circ V} \mathcal{E}_N = U \circ V,$$

for any $U, V \in TN$.

However M is a Frobenius manifold, so we have

$$\bar{\nabla}_U(V \circ \mathcal{E}) - (\bar{\nabla}_U V) \circ \mathcal{E} + U \circ \bar{\nabla}_V \mathcal{E} - \bar{\nabla}_{U \circ V} \mathcal{E} = U \circ V,$$

for any $U, V \in TN$.

We compute the l.h.s. of this equality and get

$$\begin{aligned} \bar{\nabla}_U(V \circ \mathcal{E}) - (\bar{\nabla}_U V) \circ \mathcal{E} + U \circ \bar{\nabla}_V \mathcal{E} - \bar{\nabla}_{U \circ V} \mathcal{E} \\ = \bar{\nabla}_U(V \circ \mathcal{E}_N) - (\bar{\nabla}_U V) \circ \mathcal{E}_N + U \circ \bar{\nabla}_V \mathcal{E}_N - \bar{\nabla}_{U \circ V} \mathcal{E}_N \\ + \bar{\nabla}_U(V \circ \mathcal{E}_N^\perp) - (\bar{\nabla}_U V) \circ \mathcal{E}_N^\perp + U \circ \bar{\nabla}_V \mathcal{E}_N^\perp - \bar{\nabla}_{U \circ V} \mathcal{E}_N^\perp \end{aligned}$$

Computing first term

$$\begin{aligned} \text{pr}(\bar{\nabla}_U(V \circ \mathcal{E}_N) - (\bar{\nabla}_U V) \circ \mathcal{E}_N + U \circ \bar{\nabla}_V \mathcal{E}_N - \bar{\nabla}_{U \circ V} \mathcal{E}_N) \\ = \nabla_U(V \circ \mathcal{E}_N) - (\nabla_U V) \circ \mathcal{E}_N + U \circ \nabla_V \mathcal{E}_N - \nabla_{U \circ V} \mathcal{E}_N \end{aligned}$$

So we just need to prove

$$\text{pr}(\bar{\nabla}_U(V \circ \mathcal{E}_N^\perp) - (\bar{\nabla}_U V) \circ \mathcal{E}_N^\perp + U \circ \bar{\nabla}_V \mathcal{E}_N^\perp - \bar{\nabla}_{U \circ V} \mathcal{E}_N^\perp) = 0.$$

computing directly we find

$$\begin{aligned}
& \text{pr}(\overline{\nabla}_U(V \circ \mathcal{E}_N^\perp) - (\overline{\nabla}_U V) \circ \mathcal{E}_N^\perp + U \circ \overline{\nabla}_V \mathcal{E}_N^\perp - \overline{\nabla}_{U \circ V} \mathcal{E}_N^\perp) \\
&= -A_{V \circ \mathcal{E}_N^\perp}(U) - \text{pr}(h(U, V) \circ \mathcal{E}_N^\perp) - U \circ A_{\mathcal{E}_N^\perp}(V) + A_{\mathcal{E}_N^\perp}(U \circ V) \\
&= -A_{V \circ \mathcal{E}_N^\perp}(U) - \text{pr}(h(U, V) \circ \mathcal{E}_N^\perp) - \lambda U \circ V + \lambda U \circ V \\
&= -A_{V \circ \mathcal{E}_N^\perp}(U) - \text{pr}(h(U, V) \circ \mathcal{E}_N^\perp).
\end{aligned}$$

The second equality holds because $A_{\mathcal{E}_N^\perp} = \lambda \cdot \text{Id}$.

Claim: $-A_{V \circ \mathcal{E}_N^\perp}(U) - \text{pr}(h(U, V) \circ \mathcal{E}_N^\perp) = 0$ for any $U, V \in TN$.

In fact,

(1°) the structure connection of M is flat because M is Frobenius manifold. So the relation (1.2) holds:

$$U \circ \overline{\nabla}_V W - V \circ \overline{\nabla}_U W - [U, V] \circ W + \overline{\nabla}_U(V \circ W) - \overline{\nabla}_V(U \circ W) = 0.$$

so the orthogonal part of this coefficient must be zero, i.e.,

$$U \circ h(V, W) - V \circ h(U, W) + h(U, V \circ W) - h(V, U \circ W) = 0.$$

i.e.,

$$U \circ h(V, W) - V \circ h(U, W) = h(V, U \circ W) - h(U, V \circ W).$$

then we get:

$$\langle U \circ h(V, W) - V \circ h(U, W), \mathcal{E}_N^\perp \rangle = \langle h(V, U \circ W) - h(U, V \circ W), \mathcal{E}_N^\perp \rangle.$$

We simply the r.h.s. of this equality

$$\begin{aligned}
\langle h(V, U \circ W) - h(U, V \circ W), \mathcal{E}_N^\perp \rangle &= \langle h(V, U \circ W), \mathcal{E}_N^\perp \rangle - \langle h(U, V \circ W), \mathcal{E}_N^\perp \rangle \\
&= \langle A_{\mathcal{E}_N^\perp}(V), W \circ U \rangle - \langle A_{\mathcal{E}_N^\perp}(U), W \circ V \rangle \\
&= \lambda \langle V, W \circ U \rangle - \lambda \langle U, W \circ V \rangle \\
&= 0.
\end{aligned}$$

So

$$\langle U \circ h(V, W), \mathcal{E}_N^\perp \rangle = \langle V \circ h(U, W), \mathcal{E}_N^\perp \rangle.$$

(2°) For any $W \in TN$, we have

$$\begin{aligned}
\langle A_{V \circ \mathcal{E}_N^\perp} U, W \rangle &= \langle \overline{\nabla}_U(V \circ \mathcal{E}_N^\perp), W \rangle \\
&= U \langle V \circ \mathcal{E}_N^\perp, W \rangle - \langle V \circ \mathcal{E}_N^\perp, \overline{\nabla}_U W \rangle \\
&= 0 - \langle V \circ \mathcal{E}_N^\perp, h(U, W) \rangle \\
&= -\langle \mathcal{E}_N^\perp, V \circ h(U, W) \rangle
\end{aligned}$$

the second equality holds because $\overline{\nabla}$ is the Levi-Civita connection of \overline{g} , the last equality holds because the product is compatible to the metric \overline{g} .

Now We consider the second term

$$\langle h(U, V) \circ \mathcal{E}_N^\perp, W \rangle = \langle \mathcal{E}_N^\perp, h(U, V) \circ W \rangle.$$

for all $U, V, W \in TN$. But in (1°) We have proved that

$$\langle U \circ h(V, W), \mathcal{E}_N^\perp \rangle = \langle V \circ h(U, W), \mathcal{E}_N^\perp \rangle.$$

So for any $Z \in TN$ we have

$$\langle h(U, V) \circ \mathcal{E}_N^\perp, W \rangle + \langle A_{V \circ \mathcal{E}_N^\perp} U, W \rangle = 0,$$

i.e., for any $U, V \in TN$ we have:

$$\text{pr}(A_{V \circ \mathcal{E}_N^\perp} U + h(U, V) \circ \mathcal{E}_N^\perp) = 0.$$

So we get

$$\mathcal{L}_{\mathcal{E}_N}(\circ) = \circ$$

F6) From Remark 0.4 applied to N , we deduce that

$$\nabla(\nabla\mathcal{E}_N) = 0.$$

So $(N, \circ, g, e_N, \mathcal{E}_N)$ is a Frobenius structure on N , i.e., N is a natural Frobenius submanifold of M . \square

Remark 1.3. (1) N is the submanifold of $(M, \bar{g}, \circ, e, \mathcal{E})$. If we assume that e and \mathcal{E} are tangent to N in Theorem 0.13, we recover Theorem 0.10, then N is a natural Frobenius submanifold.

(2) In the proof of Theorem 0.13, we deduce that any two equalities can imply the third one:

$$\begin{aligned}\bar{\nabla}\mathcal{E} + (\bar{\nabla}\mathcal{E})^* &= D \cdot \text{Id} . \\ \nabla\mathcal{E}_N + (\nabla\mathcal{E}_N)^* &= D_N \cdot \text{Id} . \\ A_{\mathcal{E}_N^\perp} &= \lambda \cdot \text{Id} .\end{aligned}$$

Proposition 1.4. *Let $(M, \bar{g}, \bar{\nabla}, \circ, e, \mathcal{E})$ be a Frobenius manifold, N is a submanifold of M such that g is nondegenerate. If*

$$TN \circ TN \subseteq TN$$

$$\bar{\nabla} = \nabla$$

then N is a natural Frobenius submanifold.

Proof of proposition 1.4. By the condition

$$TN \circ TN \subseteq TN$$

we know e_N is the unit vector field of (N, TN, \circ) . And by

$$\bar{\nabla} = \nabla$$

we get $\nabla\nabla\mathcal{E}_N = 0$ and $\nabla e_N = 0$. As in the proof of Theorem 1.1, the structure connection $\bar{\nabla}$ of N is integrable and the unit e_N is ∇ -flat.

M is a Frobenius manifold, so there exist a constant D such that $\bar{\nabla}\mathcal{E} + (\bar{\nabla}\mathcal{E})^* = D \cdot \text{Id}$, for any $X, Y \in TN$. By

$$\bar{\nabla} = \nabla$$

we have:

$$\begin{aligned}\langle \nabla_X \mathcal{E}_N, Y \rangle + \langle \nabla_Y \mathcal{E}_N, X \rangle &= \langle \bar{\nabla}_X \mathcal{E}_N, Y \rangle + \langle \bar{\nabla}_Y \mathcal{E}_N, X \rangle \\ &= D \cdot \langle X, Y \rangle\end{aligned}$$

So

$$\nabla\mathcal{E}_N + (\nabla\mathcal{E}_N)^* = D \cdot \text{Id}$$

By the equivalence between Saito structure with metric and Frobenius structure on TN , we get $(N, g, \nabla, \circ, e_N, \mathcal{E}_N)$ is Frobenius manifold, i.e., N is the natural Frobenius submanifold of M . \square

Remark 1.5. We can prove Proposition 1.4 by applying Theorem 0.13 because $\bar{\nabla} = \nabla$ implies $A_{\mathcal{E}_N^\perp} = 0$. $\bar{\nabla} = \nabla$ is also not a necessary condition for a submanifold to be a natural Frobenius submanifold.

Example 1.6 ([5]). $B_3 \longrightarrow I_2(6)$

The prepotential for the Frobenius manifold constructed from B_3 is

$$F_{B_3} = \frac{1}{2}t_1^2t_3 + \frac{1}{2}t_1t_2^2 + \frac{1}{6}t_2^3t_3 + \frac{1}{6}t_2^2t_3^3 + \frac{1}{210}t_3^7.$$

The two dimensional submanifold is given by

$$\begin{aligned} t_1 &= \tau_1 - \frac{2}{3}k_2^2\tau_2^3, \\ t_2 &= k_2\tau_2^2, \\ t_3 &= \tau_2. \end{aligned}$$

The condition required for the submanifold to be a natural Frobenius submanifold reduce to $k_2(2k_2 - 3)(-2k_2 - 1) = 0$. Thus there are three natural Frobenius submanifolds given by $k_2 = 0, -1/2, +3/2$. For $k_2 = -1/2$ or $k_2 = 3/2$, the given natural Frobenius submanifolds are not totally geodesic submanifolds.

2. FROBENIUS HYPERSURFACES

In this section, we mainly talk about the classification of the natural Frobenius hypersurfaces.

For general natural Frobenius submanifold neither $\bar{\nabla} = \nabla$ nor $\mathcal{E} = \mathcal{E}_N$ is a necessary condition. But for hypersurfaces, we get that all the natural Frobenius submanifolds satisfy either $\bar{\nabla} = \nabla$ or $\mathcal{E} = \mathcal{E}_N$.

In this section we suppose $(M, \bar{g}, \circ, e, \mathcal{E})$ is a Frobenius manifold, N is a hypersurface of M such that the restricted metric g is non-degenerate, ∇ is the Levi-Civita connection of g .

Lemma 2.1. *Let (M, \bar{g}) be a Riemannian manifold, let $\bar{\nabla}$ be the Levi-Civita connection of \bar{g} , and let N be a hypersurface of M . If there exists a $\bar{\nabla}$ -flat vector field $X \in TM$ such that $X_N := \text{pr}(X|_N)$ is ∇ -flat and X is not tangent to N , then N is a totally geodesic submanifold of M .*

Proof. From the flatness of X and X_N , we get: $A_{X^\perp} = 0$. Because the codimension of N is equal to 1, the shape operator A vanishes. So N is a totally geodesic submanifold. \square

Lemma 2.2. *If N is a Frobenius submanifold of M , and e is tangent to N , then $D = D_N$.*

Proof. If N is a Frobenius submanifold of M , and e is tangent to N , so from

$$\bar{\nabla}e = 0, \nabla e_N = 0$$

we get

$$h(X, e_N) = 0,$$

for any $X \in TN$.

From

$$\begin{aligned} \bar{\nabla}\mathcal{E} + (\bar{\nabla}\mathcal{E})^* &= D \cdot \text{Id}; \\ \nabla\mathcal{E}_N + (\nabla\mathcal{E}_N)^* &= D_N \cdot \text{Id}, \end{aligned}$$

we get:

$$\langle h(X, Y), \mathcal{E}_N^\perp \rangle = -\frac{D - D_N}{2} \cdot \langle X, Y \rangle.$$

Take $X = e_N$ we get

$$(D - D_N) \cdot \langle e_N, Y \rangle = 0,$$

for any $Y \in TN$.

But g is non-degenerate in TN , so we can choose a local vector field Y_0 such that $\langle e_N, Y_0 \rangle = 1$. So $D = D_N$. \square

Proof of Proposition 0.17(a). (2) \Rightarrow (1) By Proposition 1.4.

(1) \Rightarrow (2) If N is the natural Frobenius submanifold of M , then $TN \circ TN \subseteq TN$. Because e is not tangent to N and N is hypersurface of M , by Lemma 2.1 N is totally geodesic submanifold, i.e., $\bar{\nabla} = \nabla$. \square

Proof of Proposition 0.17(b). (2) \Rightarrow (1) If $TN \circ TN \subseteq TN$ and $\overline{\nabla} = \nabla$ then by Proposition 1.4 N is a natural Frobenius submanifold of M ; otherwise if $TN \circ TN \subseteq TN$ and \mathcal{E} is tangent to N , by Theorem 0.10, we also get N is a natural Frobenius submanifold.

(1) \Rightarrow (2) Suppose N is a natural Frobenius submanifold. Because e is tangent to N , by Lemma 2.2 we know $D = D_N$, and by

$$\begin{aligned}\overline{\nabla}\mathcal{E} + (\overline{\nabla}\mathcal{E})^* &= D \cdot \text{Id}; \\ \nabla\mathcal{E}_N + (\nabla\mathcal{E}_N)^* &= D \cdot \text{Id};\end{aligned}$$

we get:

$$\langle h(X, Y), \mathcal{E}_N^\perp \rangle = 0,$$

for any $X, Y \in TN$.

Because codimension of N is 1, g is non-degenerate, and $h(X, Y), \mathcal{E}_N^\perp \in TN^\perp$, so either $h(X, Y) = 0$ for any X, Y , or $\mathcal{E}_N^\perp = 0$. That is to say either $\overline{\nabla} = \nabla$ or \mathcal{E} is tangent to N .

N is the natural Frobenius submanifold implies $TN \circ TN \subseteq TN$. \square

Remark 2.3. Proposition 0.17 classify the natural Frobenius hypersurfaces. But this classification can not be generalized to natural Frobenius submanifolds of any dimension. We will give an example of a Frobenius submanifold of codimension two such that $e_N^\perp \neq 0$, $\overline{\nabla} \neq \nabla$ and $\mathcal{E}_N^\perp \neq 0$.

Example 2.4. In example 1.6, we get two natural Frobenius submanifolds of B_3 . Take $k_2 = -1/2$. This submanifold, denoted by N , is not a totally geodesic submanifold of B_3 .

Firstly, just as in example 0.11, we construct a Frobenius manifold $(B_3 \times \mathcal{A}, \overline{g}, \circ, e, \mathcal{E})$ such that t^1, t^2, t^3, z is the flat coordinates of $B_3 \times \mathcal{A}$. Now embedding B_3 to $B_3 \times \mathcal{A}$:

$$\iota : B_3 \longrightarrow B_3 \times \mathcal{A}, P \longmapsto (P, 1).$$

consider the image $\iota(N)$ of N as a submanifold of $B_3 \times \mathcal{A}$. It is given by

$$\begin{aligned}t_1 &= \tau_1 - \frac{1}{6}\tau_2^3, \\ t_2 &= -\frac{1}{2}\tau_2^2, \\ t_3 &= \tau_2 \\ z &= 1.\end{aligned}$$

For the Frobenius manifold $B_3 \times \mathcal{A}$, we get a natural Frobenius submanifold $N \times \{1\}$ with $e_N^\perp \neq 0$, $\overline{\nabla} \neq \nabla$ and $\mathcal{E}_N^\perp \neq 0$.

For the first case e is not tangent to N , we get some properties about $\mathcal{E} \circ$ and $\overline{\nabla}\mathcal{E}$.

Corollary 2.5. *Let $(M, \overline{g}, \circ, e, \mathcal{E})$ be a Frobenius manifold, and let N be a Frobenius hypersurfaces of M such that the restricted metric g is non-degenerate. If e is not tangent to N , then $(\mathcal{E} \circ)|_{TN}, (\overline{\nabla}\mathcal{E})|_{TN} \in \text{End}(TN)$*

Proof. From proposition 0.17 we have two relations:

$$TN \circ TN \subseteq TN, \quad \overline{\nabla} = \nabla.$$

If \mathcal{E} is tangent to N , then obviously, $(\mathcal{E} \circ)|_{TN}, (\overline{\nabla}\mathcal{E})|_{TN} \in \text{End}(TN)$. Now we suppose \mathcal{E} is not tangent to N , then there exist nonzero function f such that $f e_N^\perp = \mathcal{E}_N^\perp$. But $e_N^\perp \circ TN = 0$, so $\mathcal{E}_N^\perp \circ TN = 0$, and then $(\mathcal{E} \circ)|_{TN} \in \text{End}(TN)$. Now consider $(\overline{\nabla}\mathcal{E})|_{TN}$. For any $X \in TN$:

From $\bar{\nabla} = \nabla$ we know that the second fundamental form and shape operator vanish, i.e., $h = 0$, and $A = 0$. Moreover, from $\bar{\nabla} = \nabla$, $\bar{\nabla}e = 0$ and $\nabla e_N = 0$ we get:

$$\bar{\nabla}e_N^\perp = 0.$$

N is a hypersurface of M , so there exist a function, denoted by f , such that:

$$\mathcal{E}_N^\perp = f \cdot e_N^\perp.$$

we will show that the function f is a constant, i.e., there exists a constant $\mu \in \mathbb{C}$ such that.

$$\mathcal{E}_N^\perp = \mu \cdot e_N^\perp.$$

Claim: $\bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E} \in TN^\perp$.

In fact M is a Frobenius manifold, so we have:

$$(\bar{\nabla}_e \mathcal{E})|_N = e|_N$$

By $\bar{\nabla} = \nabla$ we can simplify this equality and get:

$$\nabla_{e_N} \mathcal{E}_N + \nabla_{e_N^\perp}^\perp \mathcal{E}_N^\perp + \bar{\nabla}_{e_N^\perp} \mathcal{E} = e_N + e_N^\perp$$

But N is a Frobenius submanifold of M , so we have $\nabla_{e_N} \mathcal{E}_N = e_N$. So we get:

$$\bar{\nabla}_{e_N^\perp} \mathcal{E} = e_N^\perp - \nabla_{e_N^\perp}^\perp \mathcal{E}_N^\perp \in TN^\perp.$$

So

$$\bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E} = f \bar{\nabla}_{e_N^\perp} \mathcal{E} \in TN^\perp.$$

The flatness of the structure connection $\bar{\nabla}$ of M implies that:

$$\bar{\nabla}(-\mathcal{E} \circ) - [\Phi, \bar{\nabla} \mathcal{E}] = -\Phi.$$

which applied to the pair of vectors (X, \mathcal{E}_N^\perp) amounts to

$$\bar{\nabla}_X(\mathcal{E}_N^\perp \circ \mathcal{E}) - \bar{\nabla}_X \mathcal{E}_N^\perp \circ \mathcal{E}|_N + X \circ \bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E} - \bar{\nabla}_{X \circ \mathcal{E}_N^\perp} \mathcal{E} = X \circ \mathcal{E}_N^\perp.$$

where $X \in TN$.

From the relation $TN \circ TN^\perp = 0$ we get

$$\bar{\nabla}_X(\mathcal{E}_N^\perp \circ \mathcal{E}_N^\perp) - \bar{\nabla}_X(\mathcal{E}_N^\perp) \circ \mathcal{E}_N^\perp + X \circ \bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E} = 0.$$

We have proved that $\bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E} \in TN^\perp$, so the above equality can be simplified to be:

$$\bar{\nabla}_X(\mathcal{E}_N^\perp \circ \mathcal{E}_N^\perp) = \bar{\nabla}(\mathcal{E}_N^\perp) \circ \mathcal{E}_N^\perp$$

By $\mathcal{E}_N^\perp = f \cdot e_N^\perp$, $\bar{\nabla}e_N^\perp = 0$ and $e_N^\perp \circ e_N^\perp = e_N^\perp$ we get

$$X(f^2) = 0.$$

So there exists a constant $\mu \in \mathbb{C}$ such that $f = \mu$, i.e., $\bar{\nabla} \mathcal{E}_N^\perp = \mu \bar{\nabla} e_N^\perp = 0$.

Then we get:

$$(\bar{\nabla} \mathcal{E})|_{TN} = \bar{\nabla} \mathcal{E}_N + \bar{\nabla} \mathcal{E}_N^\perp = \nabla \mathcal{E}_N \in \text{End}(TN). \quad \square$$

Remark 2.6. We have another way to see $\bar{\nabla} \mathcal{E}_N^\perp = 0$ in the proof of corollary 2.5:

N is a hypersurface, and \mathcal{E} is not tangent to N , so we just need to check

$$(2.7) \quad \langle \bar{\nabla}_X \mathcal{E}_N^\perp, \mathcal{E}_N^\perp \rangle = 0.$$

for any $X \in TN$.

From the relation $\bar{\nabla} \mathcal{E} + (\bar{\nabla} \mathcal{E})^* = D \cdot \text{Id}$, we get

$$\begin{aligned} \langle \bar{\nabla}_X \mathcal{E}_N^\perp, \mathcal{E}_N^\perp \rangle &= \langle \bar{\nabla}_X \mathcal{E}, \mathcal{E}_N^\perp \rangle \\ &= D \langle X, \mathcal{E}_N^\perp \rangle - \langle \bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E}, X \rangle \\ &= -\langle \bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E}, X \rangle \end{aligned}$$

where $X \in TN$.

In the proof of corollary 2.5, we have the relation

$$\bar{\nabla}_{\mathcal{E}_N^\perp} \mathcal{E} \in TN^\perp.$$

So for any $X \in TN$, we get

$$\langle \bar{\nabla}_X \mathcal{E}_N^\perp, \mathcal{E}_N^\perp \rangle = 0$$

i.e.,

$$\bar{\nabla}_X \mathcal{E}_N^\perp = 0.$$

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